Matching problem

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Basic notation and definition

Matching

Given an undirected graph $G = (V, E)$. A **matching** is a subset of edges $M \subset E$ such that for all vertices $v \in V$, at most one edge of M is incident on v. A **maximum matching** is a match of maximum cardinality.

Basic notation and definition

Bipartie graph

A **bipartie graph** is a graph which can be divided into 2 disjoint sets U and V such that every edge connects a vertex in U to a vertex in V.

Basic notation and definition

Free node

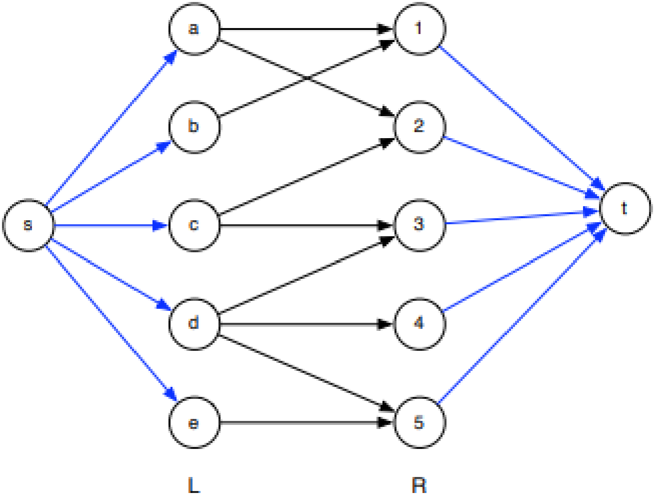
Given a match M, a node is not part of matching is called **free node**.

Matching edge

Given a match M, edges that are part of matching are called **matching edge**. Others are called **not-matching edges**.

Ford-Fulkerson method

Create a source and a sink vertex for the bipartie graph.



Hopcroft-Karp algorithm

Alternating path

Given a match M, the path in which edges belong to the matching and not-matching alternatively is called **alternating path**.

Augmenting path

An alternating path that starts from and ends on free vertices **augmenting path**.

Hopcroft-Karp algorithm

Hopcroft-Karp algorithm

- Initialize maximal matching M as empty.
- While there exists an augmenting path p, we remove matching edges of p from M and add not-matching edges of p to M.
- Return M.

Hopcroft-Karp algorithm

Finding an augmenting path

- Create a ghost vertex that connects to all left or right vertices.
- Start BFS from the ghost vertices.
- Find all vertex-disjoint shortest augmenting paths.

Hopcroft-Karp algorithm

Complexity of the algorithm

The complexity of Hocroft-Karp algorithm is $O(E\sqrt{V})$.

The stable marriage problem

Stable marriage problem (SMP)

Given n men and n women, where each person has ranked all members of the opposite sex in order of preference, marry the men and women together such that there are no two people of opposite sex who would both rather have each other than their current partners. When there are no such pairs of people, the set of marriages is deemed stable.

Gale–Shapley algorithm

Gale–Shapley algorithm

- The free man m propose to the woman w who is on top of his list and he has not yet proposed to.
  - If w is free then m and w become engaged.
  - If not then w has already engaged to a man m'. We have 2 cases:
    1. If w prefer m to m' then w and m become engaged and m' becomes free.
    2. If not then m move to another woman on the list.
- Repeat the procedure until there is no free man.

Gale–Shapley algorithm

Complexity of the algorithm

The complexity of Gale–Shapley algorithm is $O(n^2)$.

The Assignment problem

The assignment problem

The problem instance has a number of agents and a number of tasks. Any agent can be assigned to perform any task, incurring some cost that may vary depending on the agent-task assignment. It is required to perform all tasks by assigning exactly one agent to each task and exactly one task to each agent in such a way that the total cost of the assignment is minimized.

Hungarian algorithm

Hungarian algorithm

1. We first represent the input as a matrix.
2. For each row of the matrix, subtract the smallest element from every elements.
3. We do the same for each column.
4. Cover all zeros in the matrix using minimum number of horizontal and vertical lines.
    - If the minimum covering lines is n then we got an optimal assignment.
    - If not, determine the smallest entry not covered by any line.
5. Subtract this from every uncovered row and add it to each covered column. then return to step 4.

Hungarian algorithm

Complexity of the algorithm

The complexity of Hungarian algorithm is $O(n^4)$.